MTH 166

Lecture-20

Solution of Wave Equation

Topic:

Solution of Partial Differential Equations

Learning Outcomes:

1. To solve one dimensional Wave Equation

2. D’ Alembert’s solution of infinitely long wave

Problem. **Solve** $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$ **[1D-Wave Equation];** c^2 **is Diffusivity constant.**

Solution. The given one dimensional wave equation is:

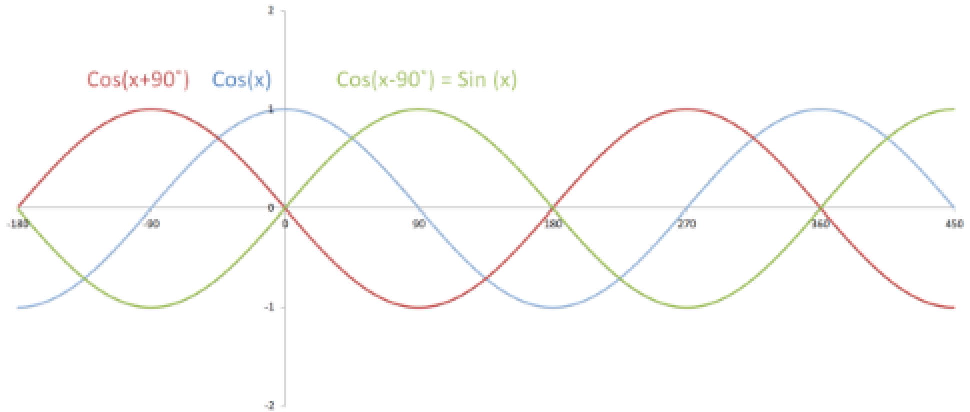
$$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)\ 0 \le x \le l\ (length), t > 0\ (time)$$

Let solution be: $u(x,t) = XT$ (2) where $X = f(x), T = g(t)$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = XT'' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

Equation (1) becomes: $XT'' = C^2 X''T$

$$\Rightarrow \frac{X''}{X} = \frac{1}{C^2}\frac{T''}{T} = k \quad \text{(Say)}$$



As k can take three values: zero, positive or negative, so we have following three cases.

Case 1. When $k = 0$

$$\frac{X''}{X} = k \qquad \Rightarrow \frac{X''}{X} = 0 \qquad \Rightarrow X'' = 0 \qquad \Rightarrow X = ax + b$$

Also $\quad \frac{1}{C^2}\frac{T''}{T} = k \qquad \Rightarrow \frac{1}{C^2}\frac{T''}{T} = 0 \qquad \Rightarrow T'' = 0 \qquad \Rightarrow T = ct + d$

Required solution of equation (1) is:

$$u(x,t) = XT = (ax + b)(ct + d)$$

Case 2. When $k = p^2$ (Positive)

$$\frac{X''}{X} = k \qquad \Rightarrow \frac{X''}{X} = p^2 \qquad \Rightarrow X'' - p^2X = 0$$

S.F. $(D^2 - p^2)X = 0$

A.E. $(D^2 - p^2) = 0 \qquad \Rightarrow D = \pm p$

$\therefore X = ae^{px} + be^{-px}$

Also $\frac{1}{C^2}\frac{T''}{T} = k \quad \Rightarrow \frac{1}{C^2}\frac{T''}{T} = p^2 \qquad \Rightarrow T'' - C^2p^2T = 0$

<u>S.F.</u> $(D^2 - C^2p^2)T = 0$

<u>A.E.</u> $(D^2 - C^2p^2) = 0 \qquad \Rightarrow D = \pm Cp$

$\therefore T = ce^{Cpt} + de^{-Cpt}$

Required solution of equation (1) is:

$u(x,t) = XT = (ae^{px} + be^{-px})(ce^{Cpt} + de^{-Cpt})$

or

$u(x,t) = XT = (a\cosh px + b\sinh px)(c\cosh Cpt + d\sinh Cpt)$

<u>Case 3.</u> When $k = -p^2$ (Negative)

$$\frac{X''}{X} = k \qquad \Rightarrow \frac{X''}{X} = -p^2 \qquad \Rightarrow X'' + p^2 X = 0$$

<u>S.F.</u> $(D^2 + p^2)X = 0$

<u>A.E.</u> $(D^2 + p^2) = 0 \qquad \Rightarrow D = \pm ip$

$\therefore X = e^{0x}(a\cos px + b\sin px)$

Also $\quad \frac{1}{C^2}\frac{T''}{T} = k \qquad \Rightarrow \frac{1}{C^2}\frac{T''}{T} = -p^2 \qquad \Rightarrow T'' + C^2 p^2 T = 0$

<u>S.F.</u> $(D^2 + C^2 p^2)T = 0$

<u>A.E.</u> $(D^2 + C^2 p^2) = 0 \qquad \Rightarrow D = \pm iCp$

$\therefore T = e^{0t}(c\cos Cpt + d\sin Cpt)$

Required solution of equation (1) is:

$$u(x,t) = XT = (a\cos px + b\sin px)\,(c\cos Cpt + d\sin Cpt)$$

This is the most suitable and practically feasible solution of wave equation.

Note: For Wave Equation

Equation: $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$

Nature: Hyperbolic

Solution: 1. $u(x,t) = (ax+b)(ct+d)$

2. $u(x,t) = (ae^{px} + be^{-px})(ce^{Cpt} + de^{-Cpt})$

or

$u(x,t) = (a\cosh px + b\sinh px)(c\cosh Cpt + d\sinh Cpt)$

3. $u(x,t) = XT = (a\cos px + b\sin px)\,(c\cos Cpt + d\sin Cpt)$ (Most suitable one)

D'Alembert's Solution of Infinitely long wave (string)

Let given wave equation be: $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$ (1)

such that $-\infty < x < \infty, t > 0$

with initial displacement $= f(x)$ and initial velocity $= g(x)$

Then. D' Alembert's solution of equation (1) is given by:

$$u(x,t) = \frac{1}{2}[f(x+Ct) + f(x-Ct)] + \frac{1}{2C}\int_{x-Ct}^{x+Ct} g(s)ds$$

Find D' Alembert's solution of following:

Problem 1. $f(x) = \sin x, g(x) = a$

Solution. $f(x+ct) = \sin(x+ct),\ f(x-ct) = \sin(x-ct), g(s) = a$

D' Alembert's solution is given by:

$$u(x,t) = \frac{1}{2}[f(x+Ct) + f(x-Ct)] + \frac{1}{2C}\int_{x-Ct}^{x+Ct} g(s)ds$$

$$\Rightarrow u(x,t) = \frac{1}{2}[sin(x+ct) + sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} a\,ds$$

$$\Rightarrow u(x,t) = \frac{1}{2}[sinxcosct + cosxsinct + sinxcosct - cosxsinct] + \frac{a}{2c}[x+ct-x+c\,t]$$

$$\Rightarrow u(x,t) = \frac{1}{2}[2sinxcosct] + \frac{a}{2c}[2ct]$$

$\Rightarrow u(x,t) = \sin x \cos ct + at$ **Answer.**

Find D' Alembert's solution of following:

Problem 2. $f(x) = 0, g(x) = \cos x$

Solution. $f(x + ct) = 0,\ f(x - ct) = 0, \quad g(s) = \cos s$

D' Alembert's solution is given by:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2C}\int_{x-ct}^{x+ct} g(s)ds$$

$$\Rightarrow u(x,t) = \frac{1}{2}[0+0] + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos s \ \ ds$$

$$\Rightarrow u(x,t) = \frac{1}{2c}[\sin s]_{x-ct}^{x+ct} = \frac{1}{2c}[\sin(x+ct) - \sin(x-ct)]$$

$$\Rightarrow u(x,t) = \frac{1}{2c}[sinxcosct + cosxsinct - sinxcosct + cosxsinct]$$

$$\Rightarrow u(x,t) = \frac{1}{c}[cosxsinct]$$ **Answer.**